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SAINT-VENANT'S PRINCIPLE IN THE CASE OF THE LOW-FREQUENCY OSCILLATIONS OF A HALF-STRIP†

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The problem of the propagation of low-frequency harmonic waves in an elastic half-strip when they are excited from one end is considered. The conditions imposed on the external actions, satisfaction of which ensures that the principle part of the solution decays asymptotically, are formulated. The results obtained can be considered as an analogue of Saint-Venant's principle in the case of the low-frequency oscillations of a half-strip.

Saint-Venant [1], on the basis of heuristic considerations, formulated the "principle of the elastic equivalence of statically equivalent forces" [2]. This principle provided the possibility of determining the position of the "Saint Venant solution" in the exact solution of the three-dimensional problem for a prismatic body with a stress-free side surface, and, in essence, justified the use of the semi-inverse method. Later, Boussinesq gave a more general formulation of this principle [2]. For massive bodies, loaded over small areas, according to Saint-Venant's principle the behaviour of the stress–strain state, at distances considerably exceeding the dimensions of an area, is determined by the principal vector and the principal moment, which, in particular, follows from an asymptotic analysis of the solution for a half-space, constructed for the first time by Boussinesq. In the 20th century a number of attempts were made to give a mathematically rigorous proof of this principle. A review of these investigations can be found in [3–6].

It has been shown strictly mathematically [7–10], that in the case of a static deformation of prismatic bodies (including a half-strip), Saint-Venant's solutions are identically equal to zero, if the principal vector and the principal moment of the stresses of the external forces applied to the ends are equal to zero. In this case the stress–strain state, generated by a self-balancing load, decreases exponentially with distance from the end. However, the exponent depends on the nature of the distribution of the self-balancing load over the end, the geometry of the cross section and the physical-mechanical properties (the non-uniformity and anisotropy) of the material. In certain cases a combination of these principles leads to the occurrence of weakly decaying solutions. In such cases Saint-Venant's principle in the classical formulation loses its meaning. The slowly decaying solutions were called "a weak boundary layer" in [11]; particular examples of problems whose solutions contain a weak boundary layer were given in [6, 12]. Hence, for laminated or rod-type structural components, in which a weak boundary layer is possible, satisfaction of the conditions for the external load to be self-balancing turns out to be insufficient to localize the stress–strain state in the neighbourhood of the region where the load is applied.

In the problem of the propagation of harmonic waves in a half-strip, homogeneous (non-decaying) modes, which are determined by the real roots of the well-known dispersion equations [13, 14], can serve as an analogue of the Saint-Venant solution. At any fixed frequency there is a finite number of such modes. The remaining modes are defined by complex roots and decay exponentially. Hence, the following question arises: what integral conditions must the amplitudes of the normal and shear stresses, specified at the end, satisfy so as to localize the oscillations at this end?

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The problem of boundary resonance, excited by a wave travelling from infinity, investigated in detail in [14], is close in character to the problem of the stress–strain state; there is also a brief review there of publications devoted to an investigation of the propagation of waves in an elastic half-space, the majority of which are devoted to this problem.

In this paper the problem of localizing the oscillations is investigated in the case when low-frequency oscillations are excited. Particular attention is given to investigating the asymptotic behaviour of the solution with respect to a dimensionless frequency parameter in the case when the amplitudes of the bending moment, and the longitudinal and shearing forces are equal to zero. The proposed consideration is based on the method of homogeneous solutions, asymptotic methods and the relations of generalized orthogonality.

1. FORMULATION OF THE PROBLEM. HOMOGENEOUS ELEMENTARY SOLUTIONS.

We will consider the problem of the propagation of harmonic waves in an elastic half-space. We will assume that any field characteristic (displacements, stresses, etc.) are proportional to $e^{-i\omega t}$, where ω is the angular frequency.

We connect the origin of a Cartesian system of coordinates Ox_1x_2 with the end of the half-strip so that $0 \le x_1 < \infty$, $-h \le x_2 \le h$. We will assume that the following boundary conditions are specified at the end $x_1 = 0$

$$\sigma_{11} = \mu p_1(x_2), \quad \sigma_{12} = \mu p_2(x_2) \tag{1.1}$$

while the front surfaces are stress-free

$$x_2 = \pm h: \sigma_{12} = 0, \quad \sigma_{22} = 0 \tag{1.2}$$

Here and below σ_{ij} are the amplitudes of the stresses and u_j are the amplitudes of the displacements.

We will introduce dimensionless coordinates $x = x_1/h, y = x_2/h$ and the displacement amplitude vector $\mathbf{u} = [u_1, u_2]^T$.

We will write the equations of the harmonic oscillations of an elastic isotropic medium in the form

$$L(-i\partial_x, \lambda)\mathbf{u} \equiv \partial_x^2 C\mathbf{u} + \partial_x B\mathbf{u} + A\mathbf{u} + \kappa^2\lambda^2\mathbf{u} = 0 \tag{1.3}$$

$$C = \begin{Vmatrix} 1 & 0 \\ 0 & \kappa^2 \end{Vmatrix}, \quad B = \begin{Vmatrix} 0 & (1-\kappa^2)\partial_y \\ (1-\kappa^2)\partial_y & 0 \end{Vmatrix}, \quad A = \begin{Vmatrix} \kappa^2\partial_y^2 & 0 \\ 0 & \partial_y^2 \end{Vmatrix}$$

$$\lambda = \frac{h\omega}{c_2}, \quad \kappa^2 = \frac{c_2^2}{c_1^2} = \frac{1-2\nu}{2(1-\nu)}, \quad c_2^2 = \frac{\mu}{\rho}, \quad \partial_x = \frac{\partial}{\partial x}, \quad \partial_y = \frac{\partial}{\partial y}$$

Here c_1 and c_2 are the velocities of the longitudinal and transverse waves, respectively, ν is Poisson's ratio, μ is the shear modulus and ρ is the density of the material.

Boundary conditions (1.2) take the form

$$y = \pm 1: M(-i\partial_x)\mathbf{u} \equiv (\partial_x B_g\mathbf{u} + A_g\mathbf{u}) = 0 \tag{1.4}$$

$$B_g = \begin{Vmatrix} 0 & \kappa^2 \\ 1-\kappa^2 & 0 \end{Vmatrix}, \quad A_g = \begin{Vmatrix} \kappa^2\partial_y & 0 \\ 0 & \partial_y \end{Vmatrix}$$

We will seek the solution of problem (1.3), (1.4) in the form

$$\mathbf{u} = h\mathbf{a}(y)e^{i\gamma x} \tag{1.5}$$

We substitute (1.5) into (1.3), (1.4), and we obtain a two-parametric eigenvalue problem on the section

$$L(\gamma, \lambda)\mathbf{a} \equiv (-\gamma^2 C + i\gamma B + A + \kappa^2\lambda^2 I)\mathbf{a} = 0 \tag{1.6}$$

$$y = \pm 1:\ M(\gamma)\mathbf{a} \equiv (i\gamma B_g + A_g)\mathbf{a} = 0 \tag{1.7}$$

Here I is the 2×2 identity matrix.

The conditions for a non-trivial solution of problem (1.6), (1.7) to exist reduce to finding the roots of the following dispersion equations:

in the case of the antisymmetric problem (Problem A)

$$D^a(\gamma, \lambda) \equiv (\lambda^2 - 2\gamma^2)^2 \cos\chi_1 \frac{\sin\chi_2}{\chi_2} + 4\gamma^2\chi_1 \sin\chi_1 \cos\chi_2 \tag{1.8}$$

in the case of the symmetric problem (Problem B)

$$D^s(\gamma, \lambda) \equiv (\lambda^2 - 2\gamma^2)^2 \cos\chi_2 \frac{\sin\chi_1}{\chi_1} + 4\gamma^2\chi_2 \sin\chi_2 \cos\chi_1 \tag{1.9}$$

where $\chi_1^2 = \lambda^2 - \gamma^2$, $\chi_2^2 = \lambda^2\kappa^2 - \gamma^2$.

For any fixed λ, Eqs (1.8) and (1.9) have a denumerable set of roots $\{\gamma_m = \gamma_m(\lambda)\}$. A normal wave of the form

$$\mathbf{U}_m = h\mathbf{u}_m e^{-i\omega t} = h\mathbf{a}_m e^{i(\gamma_m x - \omega t)}, \quad \mathbf{a}_m = [a_{1m}, a_{2m}]^T \tag{1.10}$$

corresponds to each simple root.

In the case of Problem A

$$\begin{aligned} a_{1m}^a &= A_m i\gamma[-(\lambda^2 - 2\gamma^2)\cos\chi_1 \sin(\chi_2 y) + 2\chi_1\chi_2\cos\chi_2 \sin(\chi_1 y)] \\ a_{2m}^a &= -A_m\chi_2[(\lambda^2 - 2\gamma^2)\cos\chi_1 \cos(\chi_2 y) + 2\gamma^2\cos\chi_2 \cos(\chi_1 y)] \end{aligned} \tag{1.11}$$

In the case of Problem B

$$\begin{aligned} a_{1m}^s &= B_m i\gamma[-(\lambda^2 - 2\gamma^2)\sin\chi_1 \cos(\chi_2 y) + 2\chi_1\chi_2\sin\chi_2 \cos(\chi_1 y)] \\ a_{2m}^s &= B_m\chi_2[(\lambda^2 - 2\gamma^2)\sin\chi_1 \sin(\chi_2 y) + 2\gamma^2\sin\chi_2 \sin(\chi_1 y)] \end{aligned} \tag{1.12}$$

Here A_m and B_m are normalizing factors.

We similarly have for the amplitude of the stress vector

$$\boldsymbol{\sigma}_m = \mathbf{b}_m e^{i\gamma_m x}, \quad \mathbf{b}_m = [b_{1m}, b_{2m}]^T$$

In the case of Problem A

$$\begin{aligned} b_{1m}^a &= A_m[(\lambda^2 - 2\gamma^2)(\lambda^2 - 2\chi_2^2)\cos\chi_1 \sin(\chi_2 y) - 4\gamma^2\chi_1\chi_2\cos\chi_2 \sin(\chi_1 y)] \\ b_{2m}^a &= 2A_m i\gamma\chi_2(\lambda^2 - 2\gamma^2)[\cos\chi_2 \cos(\chi_1 y) - \cos\chi_1 \cos(\chi_2 y)] \end{aligned} \tag{1.13}$$

In the case of Problem B

$$\begin{aligned} b_{1m}^s &= B_m[(\lambda^2 - 2\gamma^2)(\lambda^2 - 2\chi_2^2)\sin\chi_1 \cos(\chi_2 y) - 4\gamma^2\chi_1\chi_2\sin\chi_2 \cos(\chi_1 y)] \\ b_{2m}^s &= -2B_m i\gamma\chi_2(\lambda^2 - 2\gamma^2)[\sin\chi_1 \sin(\chi_2 y) - \sin\chi_2 \sin(\chi_1 y)] \end{aligned} \tag{1.14}$$

We will call $\mathbf{u}_m$ and $\boldsymbol{\sigma}_m$ the elementary homogeneous solutions and $\mathbf{a}_m$ and $\mathbf{b}_m$ their Spurs.

Consider the augmented vector $\mathbf{w} = [\mathbf{u}, \boldsymbol{\sigma}]^T$ and correspondingly the set of vectors $\mathbf{w}_m = \mathbf{v}_m e^{i\gamma_m x}$, where $\mathbf{v}_m = [\mathbf{a}_m, \mathbf{b}_m]^T = [a_{1m}, a_{2m}, b_{1m}, b_{2m}]^T$.

The vectors $\mathbf{a}_m$ and $\mathbf{b}_m$ are regarded as elements of a Hilbert space H with scalar product (everywhere henceforth integration is carried out over the section [–1, 1])

$$(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}) = \int (a_1^{(1)}\bar{a}_1^{(2)} + a_2^{(1)}\bar{a}_2^{(2)})dy \tag{1.15}$$

where $\bar{a}_\beta^{(j)}$ are complex-conjugate scalar functions $\bar{\mathbf{a}}^{(j)} = [\bar{a}_1^{(j)}, \bar{a}_2^{(j)}]^T$.

The augmented vectors $\mathbf{v} = [\mathbf{a}, \mathbf{b}]^T$ are regarded as elements of the Hilbert space $H_1 = H \otimes H$ with scalar products

$$(\mathbf{v}^{(1)}, \mathbf{v}^{(2)})_1 = (\mathbf{a}^{(1)}, \mathbf{a}^{(2)}) + (\mathbf{b}^{(1)}, \mathbf{b}^{(2)}) \tag{1.16}$$

The vectors $\mathbf{u}$ and $\mathbf{w}$ are regarded as the vector functions $\mathbf{u}(x)$ and $\mathbf{w}(x)$ with values in the Hilbert spaces H and H_1 respectively.

We also introduce over the vectors $\mathbf{v}$ and the vector functions $\mathbf{w}(x)$ the indefinite scalar product

$$\begin{aligned} \langle \mathbf{v}^{(1)}, \mathbf{v}^{(2)} \rangle &= (\mathbf{v}^{(1)}, J\mathbf{v}^{(2)}) = i[(a^{(1)}, b^{(2)}) - (b^{(1)}, a^{(2)})] \\ \langle \mathbf{w}^{(1)}, \mathbf{w}^{(2)} \rangle(x) &= (\mathbf{w}^{(1)}, J\mathbf{w}^{(2)})(x) \end{aligned} \tag{1.17}$$

where

$$J = i\begin{Vmatrix} 0 & -I \\ I & 0 \end{Vmatrix}$$

For the vectors $\mathbf{w}(x)$ this indefinite scalar product has a clear physical meaning, namely, the energy flux through a cross section, averaged over a period, can be expressed as follows:

$$P(\mathbf{w}) = \omega h \mu \langle \mathbf{w}, \mathbf{w} \rangle / 4 = \omega h \mu \langle \mathbf{v}, \mathbf{v} \rangle / 4 \tag{1.18}$$

2. RELATIONS OF GENERALIZED ORTHOGONALITY AND THE PROPERTIES OF THE ELEMENTARY SOLUTIONS

We denote by Λ the set of eigenvalues $\{\gamma_m\}$, by M_a the set of eigenvectors $\{\mathbf{a}_m\}$ and by M_v the set of vectors $\{\mathbf{v}_m\}$.

We will denote the real roots of Eqs (1.8) and (1.9) by $\gamma_r^{\pm} = \pm\gamma_r$ ($r = 1, \dots, N$, where $N = N(\lambda)$ is a natural number), and the eigenvectors and the Spurs of the elementary solutions corresponding to them by $\mathbf{a}_r^+, \mathbf{v}_r^+$ if $\langle \mathbf{v}_r^+, \mathbf{v}_r^+ \rangle = d_r^+ > 0$, and by $\mathbf{a}_r^-, \mathbf{v}_r^-$ if $\langle \mathbf{v}_r^-, \mathbf{v}_r^- \rangle = d_r^- < 0$.

It follows from (1.18) that $d_r^+(d_r^-)$ are proportional to the energy fluxes, while the group velocity is related to the energy flux as follows:

$$c_g = \frac{2P(\mathbf{w}_r)}{\lambda^2 c_2^2 \|\mathbf{a}_r\|^2}, \quad \|\mathbf{a}\|^2 = h\int(|a_1|^2 + |a_2|^2)dy$$

Hence, we will give a superscript plus to those elementary solutions which transfer the energy in the positive direction of the Ox_1 axis, and a superscript minus to those elementary solutions which transfer the energy in the negative direction.

We will denote the real part of the spectrum by Λ_R. Apart from real roots, Eqs (1.8) and (1.9) have denumerable sets of complex roots, arranged symmetrically in the complex plane. We will denote roots by γ_k^+ if $\operatorname{Im}\gamma_k^+ > 0$, and by γ_k^- if $\operatorname{Im}\gamma_k^- < 0$. Taking into account the symmetry of their arrangement, we will also use the notation

$$\gamma_k^+ = \gamma_k \ (\operatorname{Re}\gamma_k > 0, \operatorname{Im}\gamma_k > 0), \quad \gamma_k^- = \bar{\gamma}_k, \quad \gamma_{-k}^+ = -\bar{\gamma}_k, \quad \gamma_{-k}^- = -\gamma_k$$

We will denote the set $\{\gamma_k^{\pm}\}$ by $\Lambda_C^{\pm}$. The corresponding elementary solutions $\mathbf{u}_k^{\pm}(x)$, $\mathbf{w}_k^{\pm}(x)$ decay exponentially as $x \to \pm\infty$.

Hence

$$\Lambda = \Lambda_R \cup \Lambda_C^+ \cup \Lambda_C^-$$

and, correspondingly

$$M_{\mathbf{a}} = M_{\mathbf{a}R} \cup M_{\mathbf{a}C}^+ \cup M_{\mathbf{a}C}^-, \quad M_{\mathbf{v}} = M_{\mathbf{v}R} \cup M_{\mathbf{v}C}^+ \cup M_{\mathbf{v}C}^-$$

We will present the relations of generalized orthogonality in the form of assertions, the proof of which can be found in [15–17].

Assertion 1. We have the following relations of generalized orthogonality for the Spurs of the elementary solutions

$$\begin{aligned}
&\langle \mathbf{v}_r^\pm, \mathbf{v}_q^\pm \rangle = d_r^\pm \delta_{rq}, \quad \langle \mathbf{v}_r^\pm, \mathbf{v}_q^\mp \rangle = 0, \quad \mathbf{v}_r^\pm, \mathbf{v}_q^\pm \in M_R \\
&d_r^- = -d_r^+ = -d_r, \quad d_r > 0 \\
&\langle \mathbf{v}_r^\pm, \mathbf{v}_k^\pm \rangle = \langle \mathbf{v}_r^\pm, \mathbf{v}_k^\mp \rangle = 0, \quad \mathbf{v}_r^\pm \in M_R, \quad \mathbf{v}_k^\pm \in M_C^\pm \\
&\langle \mathbf{v}_k^\pm, \mathbf{v}_l^\mp \rangle = d_k^\pm \delta_{kl}, \quad \mathbf{v}_k^\pm, \mathbf{v}_l^\mp \in M_C^\pm; \quad d_k^- = \bar{d}_k^+, \quad d_{-k}^+ = -d_k^+
\end{aligned} \tag{2.1}$$

We will introduce the following two-component vectors and matrix

$$\mathbf{f}_m = [a_{1m}, b_{2m}]^T, \quad \mathbf{g}_m = [b_{1m}, a_{2m}]^T, \quad J_0 = \operatorname{diag}\{1, -1\}$$

Assertion 2. The following relation of generalized orthogonality exists

$$(J_0 \mathbf{f}_r^\pm, \mathbf{g}_q^\pm) = -\frac{i}{2} d_r^\pm \delta_{rq} \quad (J_0 \mathbf{f}_r^\pm, \mathbf{g}_k^\pm) = 0 \quad (J_0 \mathbf{f}_k^\pm, \mathbf{g}_l^\mp) = -\frac{i}{2} d_k^\pm \delta_{rq} \tag{2.2}$$

Certain relations of orthogonality will be required below in the case when $\lambda = 0$ (the static problem). In this case the eigenvalues of the Rayleigh–Lamb equations (1.8) and (1.9) degenerate respectively to the following

$$D_0^a(\gamma) = \gamma(\operatorname{sh} 2\gamma - 2\gamma) = 0 \tag{2.3}$$

$$D_0^s(\gamma) = \gamma(\operatorname{sh} 2\gamma + 2\gamma) = 0 \tag{2.4}$$

and the real part of the spectrum degenerates into a sextuple eigenvalue $\gamma_0 = 0$, to which two Jordan chains correspond.

In the case of Problem *A* the Jordan chain consists of the eigenvector

$$\mathbf{a}_0^a = [0, 1]^T \tag{2.5}$$

and three associated vectors

$$\mathbf{a}_1^a = [-iy, 0]^T, \quad \mathbf{a}_2^a = [0, \psi(y)]^T, \quad \mathbf{a}_3^a = [-i\theta(y), 0]^T \tag{2.6}$$

where

$$\psi = -\frac{\nu y^2}{2(1-\nu)} - \frac{9 - 13\nu - \nu^2}{30(1-\nu)^2}, \quad \theta = y\left[\frac{(2-\nu)y^2}{6(1-\nu)} + \frac{-39 + 43\nu + \nu^2}{30(1-\nu)^2}\right]$$

In the case of Problem B the Jordan chain consists of one eigenvector and one associated vector

$$\mathbf{a}_0^s = [1, 0]^T, \quad \mathbf{a}_1^s = \left[0, -\frac{i\nu y}{1-\nu}\right]^T \tag{2.7}$$

In this case

$$\mathbf{b}_0^a = \mathbf{b}_1^a = 0, \quad \mathbf{b}_2^a = \left[\frac{2y}{1-\nu}, 0\right]^T, \quad \mathbf{b}_3^a = \left[0, \frac{i(1-y^2)}{1-\nu}\right]^T; \quad \mathbf{b}_0^s = 0, \quad \mathbf{b}_1^s = \left[\frac{2i}{1-\nu}, 0\right]^T \tag{2.8}$$

Similar Jordan chains were obtained previously in [6, 10, 18] for a cylinder with an arbitrary cross-section, and a method of constructing them was given.

Assertion 3. When $\lambda = 0$ for any eigenvalue $\gamma_k^{\pm} \in \Lambda_C^{\pm}$ we have the following relations

$$(\mathbf{b}_k^{\pm}, \mathbf{a}_0^a) = 0 \quad (\mathbf{b}_k^{\pm}, \mathbf{a}_1^a) = 0 \quad (\mathbf{b}_k^{\pm}, \mathbf{a}_0^s) = 0 \tag{2.9}$$

One more assertion follows from the theorems proved in [15, 18].

Assertion 4. When $\lambda > 0$, the systems of vectors M_{aC}^+, M_{aC}^- are minimum and complete in the space H.

Assertion 5. When $\lambda = 0$ the systems of vectors $M_0^+ = \{\mathbf{a}_0^a, \mathbf{a}_1^a, \mathbf{a}_0^s, \mathbf{a}_{0k}^+\}$, $M_0^- = \{\mathbf{a}_0^a, \mathbf{a}_1^a, \mathbf{a}_0^s, \mathbf{a}_{0k}^-\}$ are minimum and complete in the space H. In the reduced sets $\mathbf{a}_{0k}^{\pm}$ are the Spurs of the elementary solutions of the static problem, corresponding to the complex eigenvalues.

From these assertions we obtain the following.

Assertion 6. Any solution of Eq. (1.3), while satisfies boundary conditions (1.4) and the condition of energy radiation, can be represented in the form

$$\mathbf{u} = \sum_r C_r \mathbf{u}_r^+(x) + \sum_k C_k \mathbf{u}_k^+(x) \tag{2.10}$$

where C_r and C_k are arbitrary constants.

From relations (2.10) we obtain the following representations

$$\boldsymbol{\sigma} = \sum_r C_r \boldsymbol{\sigma}_r^+(x) + \sum_k C_k \boldsymbol{\sigma}_k^+(x), \quad \mathbf{w} = \sum_r C_r \mathbf{w}_r^+(x) + \sum_k C_k \mathbf{w}_k^+(x) \tag{2.11}$$

In relations (2.10), (2.11) and everywhere henceforth the summation over r is carried out for all r corresponding to the roots γ_r^+, while summation over k is carried out for all k corresponding to the roots γ_k^+.

3. DETERMINATION OF THE EXPANSION COEFFICIENTS

We will introduce the following notation

$$\mathbf{u}(0) = \mathbf{u}^0 = [u_1^0, u_2^0]^T \tag{3.1}$$

$$\boldsymbol{\sigma}(0) = \boldsymbol{\sigma}^0 = [p_1, p_2]^T \tag{3.2}$$

Our further discussion will be based on the solution of two simpler problems for a half-strip with mixed boundary conditions.

The first problem. Suppose the following conditions are specified for $x = 0$

$$u_1(0) = u_1^0, \quad \sigma_{12}(0) = p_2 \tag{3.3}$$

We will represent the vector $\mathbf{f}(x) = [u_1(x), \sigma_{12}(x)]^T$ in the form of a series in elementary solutions. We have

$$\mathbf{f}(x) = \sum_r C_r^+ \mathbf{f}_r^+ e^{i\gamma_r^+ x} + \sum_r C_k^+ \mathbf{f}_k^+ e^{i\gamma_k^+ x}$$

Assuming $x = 0$, we obtain the functional relation

$$\sum_r C_r^+ \mathbf{f}_r^+ + \sum_r C_k^+ \mathbf{f}_k^+ = \mathbf{f}^0 = [u_1^0, p_2]^T$$

multiplying which in succession from the right by the vectors $J_0 \mathbf{g}_q^+$ and $J_0 \mathbf{g}_l^-$, taking relations (2.2) into account, we obtain

$$-\frac{1}{2} i d_q^+ C_q^+ = \int (u_1^0 \overline{b_{1q}^+} - p_2 \overline{a_{2q}^+}) dy, \quad q = 1, \ldots, N(\lambda); \quad -\frac{1}{2} i d_l^+ C_l^+ = \int (u_1^0 \overline{b_{1l}^-} - p_2 \overline{a_{2l}^-}) dy \qquad (3.4)$$

Here and below the subscript l takes all values corresponding to the complex roots from the set Λ_C^-.

The second problem. Suppose we are given the following conditions when $x = 0$

$$u_2(0) = u_2^0, \quad \sigma_{11}(0) = p_1 \qquad (3.5)$$

We introduce the vector $\mathbf{g}(x) = [\sigma_{11}(x), u_2(x)]^T$. As before, multiplying the corresponding relation in succession from the left by the set of vectors $J_0 \mathbf{f}_q^+, J_0 \mathbf{f}_l^-$, we obtain

$$\frac{1}{2} i d_q^+ C_q^+ = \int (p_1 \overline{a_{1q}^+} - u_2^0 \overline{b_{2q}^+}) dy, \quad \frac{1}{2} i d_l^+ C_l = \int (p_1 \overline{a_{1l}^-} - u_2^0 \overline{b_{2l}^-}) dy \qquad (3.6)$$

Hence, for both types of boundary conditions we obtain exact integral representations for the expansion coefficients.

Consider the problem with original boundary conditions (3.2). In this case the boundary-value problem can be reduced to an infinite algebraic system in the expansion coefficients. A universal method of constructing such systems was described in [14]. However, we will use another method here, as a result of which a system is obtained which is more convenient for further analysis in the case of small λ.

We will transform the functional boundary condition

$$\sum_r C_r^+ \mathbf{b}_r^+ + \sum_k C_k^+ \mathbf{b}_k^+ = \boldsymbol{\sigma}^0 \qquad (3.7)$$

into infinite algebraic systems for the antisymmetric and symmetric problems separately.

We first note that in the case of Problem A, $p_1 = p_1^a(y)$ is an odd function and $p_2 = p_2^a(y)$ is an even function. In the case of Problem B, $p_1 = p_1^s(y)$ is an even function and $p_2 = p_2^s(y)$ is an odd function.

On the basis of Assertion 5, we introduce the following systems of vectors

$$M_0^a = \{\mathbf{a}_0^a, \mathbf{a}_1^a, \mathbf{a}_l^{0a}\}, \quad M_0^s = \{\mathbf{a}_0^s, \mathbf{a}_l^{0s}\}$$

which will be complete and minimum in the space H on the set of vectors whose components have corresponding symmetry with respect to the variable y.

Multiplying Eq. (3.7) successively by the elements of the system M_0^a, we obtain

$$\sum_r d_{1r}^a C_r^a + \sum_k d_{1k}^a C_k^a = q_1^a, \quad \sum_r d_{2r}^a C_r^a + \sum_k d_{2k}^a C_k^a = q_2^a$$
$$\sum_r d_{lr}^a C_r^a + \sum_k d_{lk}^a C_k^a = q_l^a \qquad (3.8)$$

Here

$$d_{1r}^a = (\mathbf{b}_r^{a+}, \mathbf{a}_0^a) = \int b_{2r}^{a+} dy, \quad d_{1k}^a = (\mathbf{b}_k^{a+}, \mathbf{a}_0^a) = \int b_{2k}^{a+} dy$$
$$d_{2r}^a = (\mathbf{b}_r^{a+}, \mathbf{a}_1^a) = i \int b_{1r}^{a+} y dy, \quad d_{2k}^a = (\mathbf{b}_k^{a+}, \mathbf{a}_1^a) = i \int b_{1k}^{a+} y dy \qquad (3.9)$$

$$d^a_{lr} = (\mathbf{b}^{a+}_r, \mathbf{a}^{0a}_l) = \int (b^{a+}_{1r}\overline{a^{0a}_{1l}} + b^{a+}_{2r}\overline{a^{0a}_{2l}})dy, \quad d^a_{lk} = (\mathbf{b}^{a+}_k, \mathbf{a}^{0a}_l) = \int (b^{a+}_{1k}\overline{a^{0a}_{1l}} + b^{a+}_{2k}\overline{a^{0a}_{2l}})dy$$
$$q^a_1 = \int p^a_2 dy, \quad q^a_2 = i\int p^a_1 y dy, \quad q^a_l = \int (p^a_1\overline{a^{0a}_{1l}} + p^a_2\overline{a^{0a}_{2l}})dy \tag{3.10}$$

Carrying out similar transformations using the system M^s_0, we obtain

$$\sum_r d^s_{1r}C^s_r + \sum_k d^s_{1k}C^s_k = q^s_1, \quad \sum_r d^s_{lr}C^s_r + \sum_k d^s_{lk}C^s_k = q^s_l \tag{3.11}$$

where

$$d^s_{1r} = \int b^{s+}_{1r}dy, \quad d^s_{1k} = \int b^{s+}_{1k}dy$$
$$d^s_{lr} = \int (b^{s+}_{1r}\overline{a^{0s}_{1l}} + b^{s+}_{2r}\overline{a^{0s}_{2l}})dy, \quad d^s_{lk} = \int (b^{s+}_{1k}\overline{a^{0s}_{1l}} + b^{s+}_{2k}\overline{a^{0s}_{2l}})dy \tag{3.12}$$
$$q^s_1 = \int p^s_1 dy, \quad q^s_l = \int (p^s_1\overline{a^{0s}_{1l}} + p^s_2\overline{a^{0s}_{2l}})dy$$

4. ANALYSIS OF THE ANTISYMMETRIC PROBLEM AT LOW FREQUENCIES

We will first investigate the roots of Eq. (1.8) for small values of the parameter λ. As mentioned above, when $\lambda = 0$ the degenerate equation (2.3) has a quadruple root $\gamma_0 = 0$ and a denumerable set of roots $\alpha_m = \lim_{\lambda\to 0}\gamma_m(\lambda)$, which keep the same structure of the distribution in the complex plane as γ_m.

We will dwell initially on the investigation of the structure of the spectrum in the neighbourhood of γ_0 for small λ. We expand the left-hand side of Eq. (1.8) in series in powers of γ and λ, and we obtain an approximate dispersion equation

$$15(1-\nu)\lambda^2 - 10\alpha^4 - 2\alpha^2(\alpha^4 - 5(2-\nu)\lambda^2) = 0$$

Seeking its solution in the form

$$\alpha = t_0^{1/4}(\lambda^{1/2} + t_1\lambda^{3/2} + \dots)$$

we find that, in the neighbourhood of $\lambda = 0$, there are four roots

$$\alpha_1^+ = i\zeta, \quad \alpha_1^- = -i\zeta, \quad \alpha_2^+ = \eta, \quad \alpha_2^- = -\eta$$

where

$$\zeta = \lambda^{1/2}t_0^{1/4}(1-\lambda t_1), \quad \eta = \lambda^{1/2}t_0^{1/4}(1+\lambda t_1); \quad t_0 = \frac{3}{2}(1-\nu), \quad t_1 = \frac{17-7\nu}{20\sqrt{6(1-\nu)}}$$

Using perturbation theory to investigate the remaining roots of the equation, we obtain analytical expansions of the form

$$\alpha_m = \gamma_m + O(\lambda^2)$$

where γ_m are the roots of Eq. (2.3).

Analysis of Eq. (1.8) enables us to assert that, for small λ, there are only two non-decaying elementary solutions ($N(\lambda) = 1$)

$$\mathbf{u}_1^{\pm} = \mathbf{a}_1^{\pm}e^{\pm i\zeta x} \tag{4.1}$$

Among the decaying elementary solutions we distinguish two

$$\mathbf{u}_2^{\pm} = \mathbf{a}_2^{\pm} e^{\pm\eta x} \tag{4.2}$$

which will be called a weak boundary layer, since for small λ the elementary solutions (4.2) decrease slightly as $x \to \pm\infty$.

The set of remaining elementary solutions will be called a strong boundary layer.

To set up the matrix of system (3.8) and then analyse it, we will present analytical expansions for the vectors $a_1^{\pm}, a_2^{\pm}, b_1^{\pm}, b_2^{\pm}$.

With an appropriate choice of the normalizing factors A_1 and A_2, we obtain, by analytical expansions of the expressions (1.11) and (1.13),

$$\begin{aligned}
\mathbf{a}_1^{\pm} &= \mathbf{a}_0^a \mp i\zeta\mathbf{a}_1^a - \zeta^2\mathbf{a}_2^a \pm i\zeta^3\mathbf{a}_3^a + O(\lambda^2)\\
\mathbf{a}_2^{\pm} &= \mathbf{a}_0^a \mp \eta\mathbf{a}_1^a + \eta^2\mathbf{a}_2^a \mp \eta^3\mathbf{a}_3^a + O(\lambda^2)\\
\mathbf{b}_1^{\pm} &= -\zeta^2\mathbf{b}_2^a \mp i\zeta^3\mathbf{b}_3^a + O(\lambda^2), \quad \mathbf{b}_2^{\pm} = \eta^2\mathbf{b}_2^a \mp \eta^3\mathbf{b}_3^a + O(\lambda^2)
\end{aligned} \tag{4.3}$$

The coefficients of expansions (4.3) are given by formulae (2.5)–(2.8).

The analytical expansions of the remaining vectors have the form

$$\mathbf{a}_k = \mathbf{a}_{0k} + O(\lambda^2), \quad \mathbf{b}_k = \mathbf{b}_{0k} + O(\lambda^2) \tag{4.4}$$

where $\mathbf{a}_{0k}$ and $\mathbf{b}_{0k}$ are the Spurs of the elementary solutions of the static problem.

Substituting expansions (4.3) and (4.4) into expression (3.9) and taking into account the relations

$$(\mathbf{b}_k^a, \mathbf{a}_0^a) = (\mathbf{b}_k^a, \mathbf{a}_1^a) = 0$$

we can obtain the following analytical expansions for the coefficients of the system

$$\begin{aligned}
d_{11}^a &= id_{12}^a = i\lambda^{3/2}d_0(1+O(\lambda)), \quad d_{21}^a = -d_{22}^a = \lambda d_0(1+O(\lambda))\\
d_{nk}^a &= \lambda^2 d_{nk}^0(1+O(\lambda^2)), \quad d_{ln}^a = \lambda d_{l1}^0(1+O(\lambda)); \quad n = 1,2\\
d_{lk}^a &= d_{lk}^0(1+O(\lambda^2)), \quad d_0 = 4i/[3(1-\nu)]
\end{aligned} \tag{4.5}$$

We substitute expressions (4.5) into system (3.8) and, using the small-parameter method, we determine the analytical relations between the coefficients C_1, C_2 and C_k and λ.

We first note that

$$q_1 = Q/(h\mu), \quad q_2 = -iM/(h^2\mu)$$

where Q is the amplitude of the shearing force and M is the amplitude of the bending moment.

When $q_1 = O(1)$, $q_2 = 0$, $q_l = O(1)$ we have, with an error $O(\lambda^2)$

$$C_1 = \frac{q_1\zeta^2}{2d_0(\eta^5 + i\zeta^5)}, \quad C_2 = \frac{\eta^2}{\zeta^2}C_1 \tag{4.6}$$

Hence, in the case considered the coefficients C_1 and C_2, which are the amplitudes of the propagating wave (the penetrating solution) and of the weak boundary layer, are respectively of the order of $\lambda^{-3/2}$. The principal terms of the coefficients C_k are of the order of $\lambda^{-1/2}$ and are determined by the solution of the infinite system

$$\sum_k d_{lk}^0 C_k = -d_{l1}^0 C_1 - d_{l2}^0 C_2 \tag{4.7}$$

When $q_1 = 0$, $q_2 = O(1)$, $q_l = O(1)$ we have, with an error $O(\lambda^2)$

$$C_1 = \frac{q_2\eta(\eta + i\zeta)}{d_0\zeta^2(\eta^2 + \zeta^2)}, \quad C_2 = -i\frac{\zeta^3}{\eta^3}C_1 \tag{4.8}$$

The principal terms of the coefficients C_k are of the order of unity and are given by the solution of the infinite system

$$\sum_k d_{lk}^0 C_k = q_l - d_{l1}^0 C_1 - d_{l2}^0 C_2 \tag{4.9}$$

It follows from relations (4.8) and (4.9) that, in the case considered, the amplitudes of the inner solution and of the weak boundary layer are of the order of λ^{-1}, while the amplitudes of the strong boundary layer are of the order of unity.

When $q_1 = 0$, $q_2 = 0$, $q_l = O(1)$, the coefficients C_1 and C_2 are of the order of λ, the coefficients C_k are of the order of unity, and the principal terms are found from the algebraic system

$$\sum_k d_{lk}^0 C_k = q_l, \quad \sum_k d_{1k}^0 C_k = 0 \tag{4.10}$$

$$d_0 C_1 - d_0 C_2 + \sum_k d_{2k}^0 C_k = 0 \tag{4.11}$$

In the case considered, the amplitudes of the displacements at the end are given by the expressions

$$u_n^0 = \sum_k C_k a_{nk}^0 + O(\lambda^2), \quad n = 1, 2$$

i.e. by a strong boundary layer, which, in turn, by Eqs (4.10), is determined by the self-balancing part of the load.

Note that, unlike the static case, the amplitude of the stresses when $q_1 = 0$ and $q_2 = 0$ do not decrease exponentially, but the ratio of the amplitudes of the penetrating solution and of the weak boundary layer to the amplitudes of the strong boundary layer will be of the order of λ^2. It is only possible to increase this order if additional requirements are imposed on the self-balancing load. It is not possible to formulate explicit conditions in terms of the specified end stresses, since the determination of C_k involves inverting infinite system (4.10).

5. ANALYSIS OF THE SYMMETRIC PROBLEM IN THE LOW-FREQUENCY RANGE

Without going into detail, we will present the main results of the analysis.

In the neighbourhood of $\gamma = 0$ Eq. (1.9) has two real roots of the form

$$\gamma_1^{\pm} = \pm\beta, \quad \beta = \lambda\sqrt{\frac{1-\nu}{2}}\left(1 + \frac{\lambda^2\nu^2}{12(1-\nu)}\right) \tag{5.1}$$

For the remaining roots we have

$$\gamma_n = \beta_n + O(\lambda^2)$$

where β_n are the roots of Eq. (2.4).

Two non-decaying elementary solutions

$$\mathbf{u}_1^{s\pm} = \mathbf{a}_1^{s\pm} e^{\pm i\beta x}, \quad \boldsymbol{\sigma}^{s\pm} = \mathbf{b}_1^{s\pm} e^{\pm i\beta x} \tag{5.2}$$

correspond to the roots γ_1, where

$$\mathbf{a}_1^{s\pm} = \mathbf{a}_0^s \pm \beta\mathbf{a}_1^s + O(\beta^2) = [1, \pm i\beta y\nu/(1-\nu)]^T + O(\beta^2)$$

$$\mathbf{b}_1^{s\pm} = \beta\mathbf{b}_1^s \pm O(\beta^3) = \pm\beta[2i/(1-\nu), 0]^T + O(\beta^3)$$

The remaining roots define a strong boundary layer.

Analysis of system (3.11) leads to the following results.

When $q_1^s \neq 0$, $q_2^s = O(1)$, $q_1^s = O(1)$ we have

$$C_1 = \beta^{-1}A_1^{(-1)} + O(\beta), \quad A_1^{(-1)} = -iq_1^s(1-\nu)/2 \tag{5.3}$$

The principal terms of the coefficients C_k are defined by the solution of the infinite system

$$\sum_k d_{lk}^0 C_k = q_l^s - d_{l1}^0 A_1^{(-1)}, \quad d_{lk}^0 = (\mathbf{b}_k^{0+}, \mathbf{a}_k^{0-})$$

The components of the vectors $\mathbf{b}_k^{0+}$, $\mathbf{a}_k^{0-}$ are obtained by taking the limit as $\lambda \to 0$ in formulae (1.12) and (1.14). Hence, in the case considered the amplitude of the propagating wave is of the order of λ^{-1}, while the amplitudes of the boundary layer elementary solutions are of the order of unity.

When $q_1^s = 0$, $q_l^s = O(1)$ we have

$$C_1 = \beta A_1^{(1)} + O(\beta^3)$$

$A_1^{(1)}$ and C_k are given by the solution of the following system

$$\sum_k d_{lk}^0 C_k = q_l^s, \quad A_1^{(1)} = \frac{i(1-\nu)}{2}\sum_k d_{lk}^0 C_k \tag{5.4}$$

The amplitudes of the components of the displacement vector when $x = 0$ are of the order of unity and their principal terms are determined by the strong boundary layer (the self-balancing part of the load).

The ratio of the amplitudes of the penetrating solution and the strong boundary layer when $q_1^s = 0$ is of the order of λ. However, unlike the antisymmetric case, it is possible here to formulate a fairly simple condition, the satisfaction of which leads to a ratio of the order of λ^3. We will derive this condition.

Suppose the conditions $p_1^s \neq 0, p_2^s = 0$ are satisfied. Then, as shown above, $u_1^0, u_2^0 = O(1)$. We return to the first formula of (3.6) ($q = 1$). Taking into account the fact that, in the case considered

$$a_{11} = 1 + \lambda^2 h_1(y), \quad b_{2q} = \lambda^4 h_2(y)$$

$$h_1(y) = -\frac{\nu}{4}y^2 - \frac{1+2\nu^2+\nu^3}{12(1-\nu^2)}, \quad h_2(y) = \frac{y(y^2-1)\nu^2}{12(1-\nu)} \tag{5.5}$$

$$d_1^s = -\lambda i\nu\sqrt{\frac{2}{1-\nu}}(1 + O(\lambda^2))$$

we obtain, after substituting expressions (5.5) and (5.6) into the first formula of (3.6)

$$\lambda\nu\sqrt{\frac{1}{2(1-\nu)}}(1 + O(\lambda^2))C_1 = \bar{q}_1 + \lambda^2\int p_1 h_1(y)dy + O(\lambda^4)$$

Hence, when the following condition holds

$$\int p_1 dy - \frac{\lambda^2\nu}{4}\int p_1 y^2 dy = 0 \tag{5.6}$$

the amplitudes of the penetrating solution will be of the order of λ^3. In particular, we can require that both integrals in condition (5.6) should be equal to zero.

6. CONCLUSIONS

It follows from the above analysis that, in problems of the dynamics of a self-balancing low-frequency boundary load, small-amplitude propagating waves can be excited. Then, if we introduce the coefficient $k = P_0/P$, where P_0 and P are the energy fluxes generated by the self-balancing and non-self-balancing loads respectively (they are proportional to the squares of the amplitudes of the propagating modes), characterizing the degree of localization of the oscillations at the end, then, in the case of the antisymmetric problem $k^a = O(\lambda^5)$ if $Q \neq 0$, and $k^a = O(\lambda^4)$ if $Q = 0$, $M \neq 0$; in the case of the symmetric problem $k^s = O(\lambda^4)$, if the second integral in condition (5.6) is not equal to zero, and $k^s = O(\lambda^6)$ if condition (5.6) is satisfied or both integrals are equal to zero.

The decay condition (5.6), obtained in explicit form, can be used directly when refining the boundary conditions in plate dynamics, including also the case when there are self-balancing loads (see, for example [19]).

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